Norms

A norm is a notion of length on a vector space.

We've already seen that if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$,

then the equation

$$\|x\|_2 = \left\{ \sum_{i=1}^{n} |x_i|^2 \right\}^{1/2}$$

gives the usual Euclidean length of x.

Definition: (norm) Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** on V is a function

$$\|\cdot\| : V \to [0, \infty)$$

such that $\forall$ $x, y, z \in V$,

1) $\|x\| = 0$ if and only if

$x = 0_V$.

2) $\|\alpha x\| = |\alpha| \, \|x\|$

$\forall$ scalars α.

3) $\|x - y\| \le \|x - z\| + \|z - y\|$

(triangle inequality)

Example 1: (p-norms)

Let $V = \mathbb{C}^n$ (or $\mathbb{R}^n$).

Let $p \in \mathbb{R}$, $p \geq 1$, and define for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$

$$\|x\|_p = \left\{ \sum_{i=1}^{n} |x_i|^p \right\}^{1/p}.$$

This is a norm for all such p.

We can also define

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

This norm is so-named

because

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$$

$\forall \ x \in \mathbb{C}^n$ and n fixed.

If $n=1$, $\|x\|_p = \|x\|_q$

for all $1 \le p, q \le \infty$.

If $n=2$, then let

$x = (1,2)$.

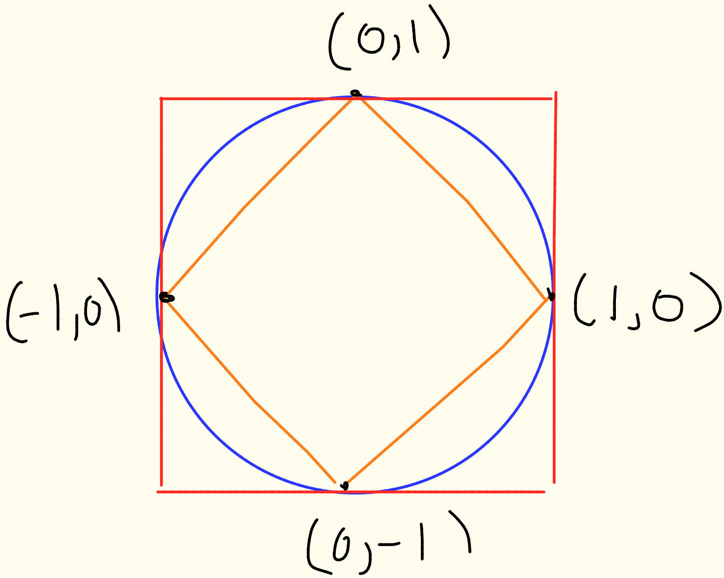
$$\|x\|_2 = \sqrt{5}$$

$$\|x\|_1 = 3$$

$$\|x\|_\infty = 2$$

all different!

Some pictures



$$\text{Circle} = \{(x,y) \in \mathbb{R}^2 \mid \|x\|_2 = 1\}$$

$$\text{Orange Square} = \{(x,y) \in \mathbb{R}^2 \mid \|x\|_1 = 1\}$$

$$\text{Red Square} = \{(x,y) \in \mathbb{R}^2 \mid \|x\|_\infty = 1\}$$

$p = 1, 2$ and ∞ are the most important norms on $\mathbb{C}^n$. One occasionally hears reference to the "0-norm" of a vector, where if $x \in \mathbb{C}^n$,

$\|x\|_0 =$ the number of nonzero coordinates of x.

This is not really a norm – why?

MatLab Calling Command

norm(v, p) gives

the p-norm of a

vector v.

Definition: (equivalent norms)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on V are equivalent if there exist positive numbers C and D with

$$\boxed{C\|x\| \leq \|x\|' \leq D\|x\|}$$

for all $x \in V$.

Example 2: One can show on $\mathbb{C}^n$ (or $\mathbb{R}^n$) that if $p \geq 1$, then

$$\|x\|_\infty \leq \|x\|_p \leq n\|x\|_\infty.$$

You can use this to show all p-norms on $\mathbb{C}^n$ are equivalent, but in fact ...

Theorem: Let V be a finite-dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$. Then all norms on V are equivalent.

proof: (kind of) We'll work only with vector spaces over $\mathbb{C}$.

Then if $\dim(V) = n$, V is isomorphic to $\mathbb{C}^n$ as a vector space over $\mathbb{C}$. So we'll further reduce to $V = \mathbb{C}^n$.

Now if $\vec{0} \neq x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$

and $\|\cdot\|$ is a norm on $\mathbb{C}^n$,

$$\|x\| = \left\| \sum_{i=1}^{n} x_i e_i \right\|$$

$$\leq \sum_{i=1}^{n} \|x_i e_i\| \quad \text{(triangle inequality)}$$

$$= \sum_{i=1}^{n} |x_i| \, \|e_i\|$$

$$\leq \sum_{i=1}^{n} \left(\max_{1 \leq i \leq n} |x_i| \right) \cdot \|e_i\|$$

$$= \|x\|_\infty \sum_{i=1}^{n} \|e_i\|$$

Then if

$$c = \frac{1}{\sum_{i=1}^{n} \|e_i\|},$$

$$c\,\|x\| \le \|x\|_\infty$$

for all $x \in \mathbb{C}^n$.

The other inequality is harder and involves some slightly sophisticated concepts to show that $\exists\, D,\ \|x\|_\infty \leq D\,\|x\|$, but it is true!

This shows any norm is equivalent to $\|\cdot\|_\infty$, which is sufficient to show that any two norms on $\mathbb{C}^n$ are equivalent. $\square$

Matrix Norms

Let A be an $m \times n$ complex matrix (book writes $A \in \mathbb{C}^{m \times n}$), $A : \mathbb{C}^n \to \mathbb{C}^m$. If $\|\cdot\|_{(n)}$ is a norm on $\mathbb{C}^n$ and $\|\cdot\|_{(m)}$ is a norm on $\mathbb{C}^m$, we can define

$$\|A\|_{(m,n)} = \max_{\|x\|_{(n)}=1} \|Ax\|_{(m)}.$$

It is a consequence of finite dimensionality that we get to write "max" instead of "sup"!

You'd have to check that $\|\cdot\|_{(m,n)}$ is actually a norm on $\mathbb{C}^{m\times n}$!

Intuitively, if $n=m$ and $\|\ \|_{(n)} = \|\ \|_{(m)}$, $\|A\|_{(m,n)}$ is the maximum amount A "strectches" the unit Sphere in the $\|\cdot\|_{(n)}$ norm.

Example 3 Let

$$A = \begin{bmatrix} 1 & 2i \\ -2i & 4 \end{bmatrix} \in \mathbb{C}^{2\times 2}$$

Note $A^* = A$. In this case,

$$\max_{\|x\|_2 = 1} \|Ax\|_2 = \text{the largest}$$

eigenvalue of A, in absolute value.

The eigenvalues of A are zero and 5,

so

$$\max_{\|x\|_2=1} \|Ax\|_2 = 5.$$

Notation: If $A \in \mathbb{C}^{m \times n}$ and we give $\mathbb{C}^n$ and $\mathbb{C}^m$ their respective p-norms for the same p, $1 \le p \le \infty$, let $\|A\|_p$ denote the norm of A as a map from $\mathbb{C}^n$ with $\|\cdot\|_p$ to $\mathbb{C}^m$ with $\|\cdot\|_p$.

Theorem: Let $A \in \mathbb{C}^{m \times n}$.

Write $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

for $a_j \in \mathbb{C}^m$, $1 \le j \le n$.

1) $\|A\|_1 = \max\limits_{1 \le j \le n} \|a_i\|_1$

2) $\|A\|_\infty = \|A^*\|_1$

3) $\|A\|_2 =$ the positive square root of the largest eigenvalue of A^*A

Proof:

1) See the book!

2) If $x \in \mathbb{C}^n$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

then if

$$1 = \|x\|_\infty = \max_{1 \leq j \leq n} |x_i|,$$

we have

$$|x_j| \leq 1 \quad \forall\ 1 \leq j \leq n.$$

$$\|Ax\|_\infty$$

$$= \max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{i,j} x_j \right|$$

$$\le \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{i,j} x_j|$$

$$= \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{i,j}| |x_j|$$

$$= \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{i,j}|$$

since $|x_j| \le 1$ for $1 \le j \le n$

Using part 1),

$$\|A^*\|_1 = \max_{1 \le i \le m} \sum_{j=1}^{n} |\bar{a}_{i,j}|$$

$$= \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{i,j}|$$

Then from the previous page;

$\|Ax\|_\infty \le \|A^*\|_1$ for all $x \in \mathbb{C}^n$, $\|x\|_\infty = 1$.

This shows $\|A\|_\infty \le \|A^*\|_1$.

Now $|a_{i,j}| = e^{i\theta_{i,j}} a_{i,j}$

for some $\theta_{i,j} \in [0, 2\pi)$,

$1 \le i \le m$, $1 \le j \le n$.

For a fixed k, $1 \le k \le m$, let

$$X_k = \begin{bmatrix} e^{i\theta_{k,1}} \\ e^{i\theta_{k,2}} \\ \vdots \\ e^{i\theta_{k,n}} \end{bmatrix}.$$

We have $\|x_k\|_\infty = 1$

$\forall \ 1 \leq k \leq m$ since

$$|e^{i\theta}| = 1 \quad \forall \ \theta \in [0, 2\pi),$$

and

$$\|Ax_k\|_\infty = \max_{1 \leq i \leq m} \left| \sum_{j=1}^{n} a_{i,j} e^{i\theta_{k,j}} \right|$$

$$\geq \left| \sum_{j=1}^{n} a_{k,j} e^{i\theta_{k,j}} \right|$$

$$= \sum_{j=1}^{n} |a_{k,j}|$$

Then

$$\|A\|_\infty \geq \max_{1 \leq k \leq m} \|Ax_k\|_\infty$$

$$\geq \max_{1 \leq k \leq m} \sum_{j=1}^{n} |a_{k,j}|$$

$$= \|A^*\|_1 .$$

We've then shown both $\|A\|_\infty \leq \|A^*\|_1$ and $\|A^*\|_1 \leq \|A\|_\infty$, so $\|A\|_\infty = \|A^*\|_1$.

3) Singular Value Decomposition

We'll need to wait a bit for this.

Example 4 Let $A = \begin{bmatrix} 2+i & 3 \\ -4 & 1-i \end{bmatrix}$.

Then $\|A\|_1 = 4+\sqrt{5}$,

$$\|A\|_\infty = 4+\sqrt{2}.$$

We have

$$A^*A = \begin{bmatrix} 21 & 2+i \\ 2-i & 11 \end{bmatrix}$$

and the eigenvalues of

A^*A are $16 \pm \sqrt{30}$,

so $\|A\|_2 = \sqrt{16+\sqrt{30}}$.

≈ 4.6344